

Isotropic Landau levels of Dirac fermions in high dimensions

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We generalize the Landau levels of two-dimensional Dirac fermions to three dimensions and above with the full rotational symmetry. Similarly to the two-dimensional case, there exists a branch of zero energy Landau levels of fractional fermion modes for the massless Dirac fermions. The spectra of other Landau levels distribute symmetrically with respect to the zero energy scaling with the square root of the Landau level indices. This mechanism is a non-minimal coupling of Dirac fermions to the background fields. This high dimensional relativistic Landau level problem is a square root problem of its previous studied non-relativistic version investigated in Li and Wu [arXiv:1103.5422 (2011)].

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I. INTRODUCTION

The integer quantum Hall effect in two-dimensional (2D) electron gas arises from the quantized 2D Landau levels (LL). The non-trivial band structure topology is characterized by non-zero Chern numbers^{2,3}. Later on, quantum anomalous Hall insulators based on Bloch-wave band structures were proposed in the absence of Landau levels⁴. In recent years, the study of topological insulators (TI) in both 2D and three dimensions (3D) has become a major focus of condensed matter physics^{5–11}. TIs maintain time-reversal (TR) symmetry, and their band structures are characterized by the nontrivial Z_2 -index. As for the 3D TIs, various materials with Bloch-wave band structures have been realized and the stable helical surface modes have been detected^{12–20}. Since LL wavefunctions have explicit forms with elegant analytical properties, TIs based on high dimensional LL structures would provide a nice platform for further theoretical studies. In particular, interaction effects in the flat LLs are non-perturbative, which could lead to non-trivial many-body states in high dimensions.

The seminal work by Zhang and Hu²¹ generalizes LLs to the compact S^4 -sphere with particles coupled to the $SU(2)$ gauge potential. The isospin of particles I scales as R^2 where R is the radius of the sphere. Such a system realizes the four dimensional integer and fractional TIs. The 3D and 2D TIs can be constructed from the 4D TIs by dimensional reduction¹⁵. Further generalizations to other manifold have also been developed^{22–27}. Recently, the LLs of non-relativistic fermions have been generalized to arbitrary dimensional flat space R^D by two of the authors¹. For the simplest case of 3D, the $SU(2)$ Aharonov-Casher gauge potential replaces the role of the usual $U(1)$ vector potential. Depending on the sign of the coupling constant, the flat LLs are characterized by either positive or negative helicity. In the positive and negative helicity channels, the eigenvalues of spin-orbit coupling term $\vec{\sigma} \cdot \vec{L}$ take values of l and $-(l+1)$, respectively. Each LL contributes a branch of helical surface modes at the open boundary. When odd numbers of LLs

are fully filled, there are odd numbers of helical Fermi surfaces. Thus the system is a 3D strong topological insulator. This construction can be easily generalized to arbitrary D -dimensions by coupling the fundamental spinors to the $SO(D)$ gauge potential.

Quantized LLs of 2D Dirac fermions have also been extensively investigated in the field theory context known as the parity anomaly^{4,28–33}. This can be viewed as the square root problem of the usual 2D non-relativistic LLs. External magnetic fields induce vacuum charges with the density proportional to the field strength. The sign of the charge density is related to the sign of the fermion mass. There is an ambiguity if the Dirac fermions are massless. In this case, there appear a branch of zero energy Landau levels. Each of them contributes $\pm \frac{1}{2}$ fermion charge. It is similar to the soliton charge in the Jackiw-Rebbi model^{34,35}, which is realized in condensed matter systems of one dimensional conducting polymer³⁶. Depending on whether the zero energy Landau levels are fully occupied or empty, the vacuum charge density is $\pm \frac{1}{2\pi l^2}$ where l' is the magnetic length. In condensed matter physics, the best known example of Dirac fermions is in graphene, which realizes a pair of Dirac cones. The quantized LLs in graphene have been observed which distribute symmetrically with respect to zero energy. Their energies scale as the square root of the Landau level index. The observed Hall conductance per spin component are quantized at odd integer values, which reflects the nature of two Dirac cones in graphene for each spin component^{37–40}.

In this article, we generalize the LLs with full rotational symmetry of Dirac fermions to the three dimensional flat space and above. It is a square root problem of the high dimensional LLs investigated in Ref. [1]. Our Hamiltonian is very simple: replacing the momentum operator in the Dirac equation by the creation or annihilation phonon operators, which are complex combinations of momenta and coordinates. The LLs exhibit the same spectra as those in the 2D case but with the full rotational symmetry in D -dimensional space. Again the zero energy Landau levels are half-fermion modes. Each LL contributes to a branch of helical surface mode at open boundaries.

This paper is organized as follows. In Sect. II, after a brief review of the 2D LL Hamiltonian of Dirac fermions in graphene, we construct the 3D LL Hamiltonian of Dirac fermions. Reducing this 3D system to 2D, it gives rise to 2D quantum spin Hall Hamiltonian of Dirac fermions with LLs. In Sect. III, we further solve this 3D LL Hamiltonian of Dirac fermions, and its edge properties are discussed. For the later discussion of generalizing the 3D LL Hamiltonian to arbitrary higher dimensions, in Sect. IV, we briefly review some properties of D dimensional spherical harmonics and spinors. In Sect. V and Sect. VI, we extend the solutions of LL Hamiltonians to arbitrary odd and even dimensions, respectively. Conclusions are given in Sect. VII.

II. THE LANDAU LEVEL HAMILTONIAN OF 3D DIRAC FERMIONS

A. A Brief Review of the 2D LL Hamiltonian

Before discussing the LL problem of Dirac fermions in 3D, we briefly review the familiar 2D case^{39,40} to gain the insight on how to generalize it to high dimensions. The celebrated condensed matter system to realize 2D Dirac fermion is the monolayer of graphene^{39,40}, which possesses a pair of Dirac cones with spin degeneracy. Here for simplicity, we only consider a single 2D Dirac cone under a uniform magnetic field $B\hat{z}$. The Landau level Hamiltonian in the xy -plane reads

$$H_{2D,LL} = v_F \left\{ (p_x - \frac{e}{c}A_x)\sigma_x + (p_y - \frac{e}{c}A_y)\sigma_y \right\}, \quad (1)$$

where the Dirac fermion with momentum \vec{p} is minimally coupled to the $U(1)$ magnetic field with symmetric gauge potentials

$$A_x = -\frac{B}{2}y, \quad A_y = \frac{B}{2}x, \quad (2)$$

satisfying $\nabla \times \vec{A} = B\hat{z}$; the Fermi velocity v_F is related to the cyclotron frequency ω via the magnetic length l' as

$$l' = \sqrt{\frac{\hbar c}{eB}}, \quad v_F = \frac{l'\omega}{\sqrt{2}}. \quad (3)$$

For later convenience, we define $l_0 = \sqrt{2}l'$ which will be termed as cyclotron length below. The spectra of Eq. 1 consist of a branch of zero energy LL, and other LLs with positive and negative energies distribute symmetrically around zero energy. The energy of each LL scales as the square root of the Landau level index. It is well-known that Eq. 1 can be recast in term of creation and annihilation operators

$$H_{2D,LL} = \frac{\hbar\omega}{\sqrt{2}} \begin{bmatrix} 0 & \hat{a}_y^\dagger + i\hat{a}_x^\dagger \\ \hat{a}_y - i\hat{a}_x & 0 \end{bmatrix}, \quad (4)$$

where $\hat{a}_i (i = x, y)$ are the phonon annihilation operators along the x and y -directions, with the form as

$$\hat{a}_i = \frac{1}{\sqrt{2}} \left\{ \frac{1}{l_0} r_i + i \frac{l_0}{\hbar} p_i \right\}. \quad (5)$$

In Eq. 4, two sets of creation and annihilation operators combine with 1 and the imaginary unit i . In order to generalize to 3D, in which there exist three sets of creation and annihilation operators, we employ Pauli matrices to match them as explained below.

B. The construction of the 3D LL Hamiltonian

We define the rotationally invariant operator \hat{B} as

$$\hat{B}_{3D} = -i\sigma_i \hat{a}_i = \sigma_i \frac{1}{\sqrt{2}} \left\{ \frac{p_i l_0}{\hbar} - i \frac{r_i}{l_0} \right\}, \quad (6)$$

where the repeated index i runs over x, y and z ; \hat{a}_i is the phonon annihilation operator along the i -direction; l_0 is the cyclotron length. We design the 3D Landau level Hamiltonian of Dirac fermions as

$$H_{3D} = \frac{\hbar\omega}{2} \begin{bmatrix} 0 & B_{3D}^\dagger \\ B_{3D} & 0 \end{bmatrix}. \quad (7)$$

Eq. 7 contains the complex combination of momenta and coordinates, thus it can be viewed as the generalized Dirac equation defined in the phase space. Using the convention of α , β and γ -matrices defined as

$$\begin{aligned} \alpha_i &= \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{bmatrix}, \\ \gamma_i &= \beta \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \\ \gamma_5 &= i\gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{bmatrix}, \end{aligned}$$

Eq. 7 is represented as

$$H_{3D} = v_F \sum_{i=x,y,z} \left\{ \alpha_i p_i + \gamma_i i \hbar \frac{r_i}{l_0^2} \right\}, \quad (8)$$

where $v_F = \frac{1}{2}l_0\omega$. A mass term can be added into Eq. 8 as

$$H_{3D,ms} = \Delta \beta = \begin{pmatrix} \Delta I_{2 \times 2} & 0 \\ 0 & -\Delta I_{2 \times 2} \end{pmatrix}. \quad (9)$$

A similar Hamiltonian was studied before under the name of Dirac oscillator^{41,42}, which corresponds to Eq. 7 plus the mass term of Eq. 9 with the special relation $l_0 = \sqrt{\hbar c^2 / \Delta \omega}$. However, the relation between the solution of such a Hamiltonian to the LLs and its topological properties were not noticed before.

The corresponding Lagrangian of Eq. 8 reads

$$\mathcal{L} = \bar{\psi} \left\{ \gamma_0 i \hbar \partial_t - i v \gamma_i \hbar \partial_i \right\} \psi - v_F \hbar \bar{\psi} i \gamma_0 \gamma_i \psi F^{0i}(r), \quad (10)$$

where $F^{0i} = x_i/l_0^2$. Compared with the usual way that Dirac fermions minimally couple to the $U(1)$ gauge field, here they couple to the background field in Eq. 10 through $i\gamma_0\gamma_i$. It can be viewed as a type of non-minimal coupling, the Pauli coupling. Apparently, Eq. 7 is rotationally invariant. It is also time-reversal invariant, and the time-reversal operation T is defined as

$$T = \gamma_1\gamma_3 K = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} K, \quad (11)$$

where K represents the complex conjugation operation, and $T^2 = -1$.

C. Reduction to the 2D quantum spin Hall Hamiltonian of Dirac fermions with LLs

If we suppress the z -component part in the definition of Eq. 6, we will arrive at double copies of the usual LL problem of 2D Dirac fermions with Kramer degeneracy, which can be considered as the Z_2 -topological insulator Hamiltonian arising from LLs of 2D Dirac fermions. We define the operator \hat{B}_{2D} as $\hat{B}_{2D} = -i\sigma_x\hat{a}_x - i\sigma_y\hat{a}_y$, and the Eq. 7 reduces to

$$\begin{aligned} & \frac{\sqrt{2}}{2}\hbar\omega \begin{pmatrix} 0 & 0 & 0 & \hat{a}_y^\dagger + i\hat{a}_x^\dagger \\ 0 & 0 & -\hat{a}_y^\dagger + i\hat{a}_x^\dagger & 0 \\ 0 & -\hat{a}_y - i\hat{a}_x & 0 & 0 \\ \hat{a}_y - i\hat{a}_x & 0 & 0 & 0 \end{pmatrix}, \\ & = v_F \begin{pmatrix} 0 & 0 & 0 & p_- - A_- \\ 0 & 0 & p_+ + A_+ & 0 \\ 0 & p_- + A_- & 0 & 0 \\ p_+ - A_+ & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $p_\pm = p_x \pm ip_y$ and $A_\pm = A_x \pm iA_y$. It is reducible into a pair of 2×2 matrices as

$$H_{2D,\pm} = v_F \begin{pmatrix} 0 & p_- \pm A_- \\ p_+ \pm A_+ & 0 \end{pmatrix}, \quad (12)$$

which are time-reversal partner to each other. Thus Eq. 12 can be viewed as the quantum spin Hall Hamiltonian of 2D Dirac fermions.

A similar situation occurs in the strained graphene systems in which lattice distortions behave like a gauge field coupling. Signatures of LLs due to strains have been observed in Ref. 43. Due to the TR symmetry, the Dirac cones at two non-equivalent vertices of the Brillouin zone see gauge fields with a opposite sign to each other. Such a coupling is also spin-independent. However, the TR transformation connecting two Dirac cones satisfies $T^2 = 1$, thus LLs due to strain are not topologically protected. They are unstable under inter-valley scattering.

Eq. 12 exhibits the standard minimal coupling to the background $U(1)$ gauge field. Its solutions are well-known thus will not be repeated here. After all, there is

no non-minimal coupling in 2D. Each state of the zero energy LL is actually a half-fermion zero mode. Whether it is filled or empty contributes the fermion charge $\pm\frac{1}{2}$. As the chemical potential $\mu = 0^\pm$, magnetic field pumps vacuum charge density $\rho(r) = \pm\frac{1}{2}\frac{e}{\hbar c}B$. In the field theory context, this is an example of the parity anomaly^{4,28–33}. Our 3D version and generalizations to arbitrary dimensions exhibit similar effects as will be discussed below.

III. THE BULK SPECTRA OF THE 3D DIRAC FERMION LLS

In this section, we will present the solution of the spectra and wavefunctions of the 3D LL Hamiltonian for Dirac fermions, which can be obtained basing on the solutions of the 3D LL problem of the non-relativistic case¹. We start with a brief review of the non-relativistic case of the 3D LL problem.

A. The 3D isotropic non-relativistic LL wavefunctions

The 3D isotropic LL Hamiltonians for non-relativistic particles are just spin- $\frac{1}{2}$ fermions in the 3D harmonic oscillator plus spin-orbit coupling¹ as

$$H_{3D,\mp} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 \mp \omega \vec{L} \cdot \vec{\sigma}. \quad (13)$$

Their eigenfunctions are essentially the same as those of the 3D harmonic oscillator of spin- $\frac{1}{2}$ fermions organized in the total angular momentum eigen-basis of j, j_z as

$$\psi_{n_r, j_\pm, l, j_z}(\vec{r}) = R_{n_r, l}(r) \mathcal{Y}_{j_\pm, l, j_z}(\hat{\Omega}), \quad (14)$$

where n_r is the radial quantum number; $j_\pm = l \pm \frac{1}{2}$ represent positive and negative helicity channels, respectively; and l is the orbital angular momentum. Please note that l is not an independent variable from j_\pm . We write it explicitly in order to keep track of the orbital angular momentum. The radial wavefunction can be represented through the confluent hypergeometric functions as

$$R_{n_r, l}(r) = N_{n_r, l} \left(\frac{r}{l_0}\right)^l F(-n_r, l + \frac{3}{2}, \frac{r^2}{l_0^2}) e^{-\frac{r^2}{2l_0^2}}, \quad (15)$$

where F is the standard first kind confluent hypergeometric function,

$$\begin{aligned} F(-n_r, l + \frac{3}{2}, \frac{r^2}{l_0^2}) &= \sum_{n=0}^{\infty} \frac{\Gamma(-n_r + n)}{\Gamma(-n_r)} \frac{\Gamma(l + \frac{3}{2})}{\Gamma(l + \frac{3}{2} + n)} \\ &\times \frac{1}{\Gamma(n+1)} \left(\frac{r^2}{l_0^2}\right)^n. \end{aligned} \quad (16)$$

When n_r is a positive integer, the sum over n is cut off at n_r . The normalization factor reads as

$$N_{n_r, l} = \frac{l_0^{-\frac{3}{2}}}{\Gamma(l + \frac{3}{2})} \sqrt{\frac{2\Gamma(l + n_r + \frac{3}{2})}{\Gamma(n_r + 1)}}. \quad (17)$$

The angular part of the wavefunction is the standard spin-orbit coupled spinor spherical harmonic function, which reads as

$$\mathcal{Y}_{j\pm, l, j_z = m + \frac{1}{2}}(\hat{\Omega}) = \begin{pmatrix} \pm \sqrt{\frac{l \pm j_z + \frac{1}{2}}{2l+1}} Y_{l, m}(\hat{\Omega}) \\ \sqrt{\frac{l \mp j_z + \frac{1}{2}}{2l+1}} Y_{l, m+1}(\hat{\Omega}) \end{pmatrix}. \quad (18)$$

Depending on the sign of the spin-orbit coupling in Eq. 13, one of the two branches of positive or negative helicity states are dispersionless with respect to j , and thus are dispersionless LLs. For $H_{3D,-}$, the positive helicity states become dispersionless LLs as

$$H_{3D,-} \psi_{n_r, j_+, l, j_z} = (2n_r + \frac{3}{2}) \hbar \omega \psi_{n_r, j_+, l, j_z} \quad (19)$$

where n_r serves as Landau level index. However, the negative helicity states are dispersive whose eigen-equation reads

$$H_{3D,-} \psi_{n_r, j_-, l, j_z} = (2n_r + 2l + \frac{5}{2}) \hbar \omega \psi_{n_r, j_-, l, j_z}. \quad (20)$$

Similarly, we have the following eigen-equations for the $H_{3D,+}$ as

$$\begin{aligned} H_{3D,+} \psi_{n_r, j_-, l, j_z} &= (2n_r + \frac{1}{2}) \hbar \omega \psi_{n_r, j_-, l, j_z} \\ H_{3D,+} \psi_{n_r, j_+, l, j_z} &= (2n_r + 2l + \frac{3}{2}) \hbar \omega \psi_{n_r, j_+, l, j_z}. \end{aligned} \quad (21)$$

In this case, the negative helicity states become dispersionless LLs with respect to j , while the positive ones are dispersive.

B. 3D LL wavefunctions of Dirac fermions

Now we are ready to present the spectra and the four-component eigenfunctions of Eq. 8 for the massless case. Its square is block-diagonal, and two blocks become the non-relativistic 3D Landau level Hamiltonians with opposite signs of spin-orbit coupling studied in Ref. [1],

$$\begin{aligned} \frac{H_{3D}^2}{\frac{1}{2} \hbar \omega} &= \begin{bmatrix} H_- & 0 \\ 0 & H_+ \end{bmatrix} \\ &= \frac{p^2}{2M} + \frac{1}{2} M \omega^2 r^2 - \omega \left\{ \vec{L} \cdot \vec{\sigma} + \frac{3}{2} \hbar \right\} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \end{aligned} \quad (22)$$

where M is defined through the relation $l_0 = \sqrt{\hbar/(M\omega)}$.

Its eigenfunctions can be represented in terms of non-relativistic Landau levels of Eq. 14 as presented in Sect. III A. Eq. 7 has a conserved quantity as

$$K = \begin{bmatrix} \vec{\sigma} \cdot \vec{l} + \hbar & 0 \\ 0 & -(\vec{\sigma} \cdot \vec{l} + \hbar) \end{bmatrix}. \quad (23)$$

According to its eigenvalues, the eigenfunctions of Eq. 7 are classified as

$$\begin{aligned} K \Psi_{\pm n_r, j, j_z}^I &= (l+1) \hbar \Psi_{\pm n_r, j, j_z}^I, \\ K \Psi_{\pm n_r, j, j_z}^{II} &= -l \hbar \Psi_{\pm n_r, j, j_z}^{II}, \end{aligned} \quad (24)$$

respectively. $\Psi_{\pm n_r, j, j_z}^I$ is dispersionless with respect to j , while $\Psi_{\pm n_r, j, j_z}^{II}$ is dispersive, respectively. The dispersionless branch is solved as

$$\Psi_{\pm n_r, j, j_z}^I(\vec{r}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{n_r, j_+, l, j_z}(\vec{r}) \\ \pm i \psi_{n_r-1, j_-, l+1, j_z}(\vec{r}) \end{bmatrix} \quad (25)$$

with the energy

$$E_{\pm n_r, j, j_z} = \pm \hbar \omega \sqrt{n_r}. \quad (26)$$

Please note that the upper and lower two components of Eq. 25 possess different values of orbital angular momenta. They exhibit opposite helicities of j_{\pm} , respectively. The zeroth Landau level ($n_r = 0$) is special: only the first two components are non-zero.

On the other hand, the wavefunctions of the dispersive branch read

$$\Psi_{\pm n_r, j, j_z}^{II}(\vec{r}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mp i \psi_{n_r, j_-, l+1, j_z}(\vec{r}) \\ \psi_{n_r, j_+, l, j_z}(\vec{r}) \end{bmatrix}, \quad (27)$$

with the spectra solved as

$$E_{\pm n_r, j, j_z} = \pm \hbar \omega \sqrt{n_r + j + 1}. \quad (28)$$

These states are just discrete energy levels lying between two adjacent LLs. For simplicity, let us only consider the positive energy states. The degeneracy of these mid-gap states lying between the n -th and $(n+1)$ -th Landau levels with $n = n_r + j + \frac{1}{2}$ is finite, $n(n+1)$, due to finite combinations of n_r and j . In particular, between the zeroth LL and the LLs with $n_r = \pm 1$, these discrete states do not exist at all.

Because Eq. 8 satisfies $\beta H_{3D} \beta = -H_{3D}$, its spectra are symmetric with respect to zero energy. If the zeroth branch of Landau levels ($n_r = 0$) are occupied, each of them contribute a half-fermion charge. The vacuum charge is $\rho^{3D}(\vec{r}) = \frac{1}{2} \sum_{j, j_z} \Psi_{0; j, j_z}^\dagger(\vec{r}) \Psi_{0; j, j_z}(\vec{r})$, which are calculated as

$$\begin{aligned} \rho^{3D}(\vec{r}) &= \frac{1}{2l_0^3} \left\{ \sum_{l=0}^{\infty} \frac{l+1}{2\pi} \frac{1}{\Gamma(l+\frac{3}{2})} \left(\frac{r^2}{l_0^2} \right)^l \right\} e^{-\frac{r^2}{l_0^2}} \\ &= \frac{1}{2\pi l_0^3} \left\{ \frac{1}{\sqrt{\pi}} e^{-\frac{r^2}{l_0^2}} + \left(\frac{r}{l_0} + \frac{l_0}{2r} \right) \text{erf}\left(\frac{r}{l_0}\right) \right\}, \\ &\rightarrow \frac{r}{2\pi l_0^4}, \quad \text{as } r \rightarrow +\infty. \end{aligned} \quad (29)$$

In 2D, the induced vacuum charge density in the gapless Dirac LL problem is a constant, $\rho^{2D}(r) = \frac{1}{2\pi l_0^2} = \frac{1}{4\pi} \frac{e}{\hbar c} B$, which is known as ‘‘parity anomaly’’. However, in the 3D case, the vacuum charge density $\rho^{3D}(r)$ diverges linearly, which is dramatically different from that

in 2D. This can be easily understood in the semi-classic picture. Each Landau level with orbital angular momentum l has a classic radius $r_l = \sqrt{2l}l_0$. In 2D, between $r_l < r < r_{l+1}$, there is only one state. However, in 3D there is the $2j_+ + 1 = 2l + 2$ fold degeneracy, which is the origin of the divergence of the vacuum charge density as r approaches infinity. Generally speaking, in the case of D dimensions, the degeneracy density scales as r^{D-2} as shown in Sect. V and Sect. VI. The intrinsic difference between high- D and 2D is that the high dimensional LL problems exhibit the form of non-minimal coupling. In 2D, due to the specialty of Pauli matrices, this kind of coupling reduces back to the usual minimal coupling. In Eq. 10, the background field is actually a linear divergent electric field, not the magnetic vector potential. Eq. 29 can be viewed as a generalization of “parity anomaly” to 3D for non-minimal couplings.

Now we consider the full Hamiltonian with the mass term Eq. 9. The mass term mixes the LLs in Eq. 25 with opposite level indices $\pm n_r$ but the same values of j and j_z . The new eigenfunctions become

$$\begin{bmatrix} \Psi_{n_r, j, j_z}^{I, \prime} \\ \Psi_{-n_r, j, j_z}^{I, \prime} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \Psi_{n_r, j, j_z} \\ \Psi_{-n_r, j, j_z} \end{bmatrix}, \quad (30)$$

where $\cos^2 \theta = \frac{1}{2} \left[1 + \sqrt{n_r} / \sqrt{n_r + [\Delta / (\hbar \omega)]^2} \right]$. The spectra are

$$E_{\pm n_r, j, j_z}^{I, ms} = \pm \sqrt{n_r (\hbar \omega)^2 + \Delta^2}. \quad (31)$$

The zeroth LL $\Psi_{n_r=0, j, j_z}^I(\vec{r})$ singles out, which is not affected by the mass term. Only its energy is shifted to Δ .

C. Gapless surface modes

As shown in Eq. 29, the 3D LL system of Dirac fermions has a center, and does not have translational symmetry. Thus how to calculate its topological index remains a challenging problem. Nevertheless, we can still demonstrate its non-trivial topological properties through the solution of its gapless surface modes.

We consider the surface spectra at a spherical boundary with a large radius $R \gg l_0$. The Hamiltonian $H_{r < R}$ inside the sphere takes the massless form of Eq. 8, while $H_{r > R}$ outside takes the mass term of Eq. 9 in the limit of $|\Delta| \rightarrow \infty$. Again the square of this Hamiltonian $(H_{r < R} + H_{r > R})^2$ is just Eq. 22 subject to the open boundary condition at the radius of R . The spectra of the open surface problem of the non-relativistic 3D LL Hamiltonian have been calculated and presented in Fig. 3 in Ref. [1], in which the spectra of each Landau level remain flat for bulk states and develop upturn dispersions as increasing j near the surface. The solution to the Dirac spectra is just to take the square root. Except the zeroth LL, each of the non-relativistic LL and its surface branch

split into a pair of bulk and surface branches in the relativistic case. The relativistic spectra take the positive and negative square roots of the non-relativistic spectra, respectively. The zeroth LL branch singles out. We can only take either the positive or negative square root, but not both. Its surface spectra are upturn or downturn with respect to j depending on the sign of the vacuum mass. For the current Hamiltonian, only the first two components of the zeroth LL wavefunction are non-zero, thus it only senses the upper 2×2 diagonal block of vacuum mass in Eq. 9, thus its surface spectra are pushed upturn.

IV. REVIEW OF D-DIMENSIONAL SPHERICAL HARMONICS AND SPINORS

We will study the LL of Dirac fermions for general dimensions in the rest part of the paper. For later convenience, we present here some background knowledge of the $SO(D)$ group which can be found in standard group theory textbooks⁴⁴.

The D -dimensional spherical harmonic functions $Y_{l, \{m\}}(\hat{\Omega})$ form the representation of the $SO(D)$ group with the one-row Young pattern, where l is the number of boxes and $\{m\}$ represents a set of $D - 2$ quantum numbers of the subgroup chain from $SO(D - 1)$ down to $SO(2)$. The degeneracy of $Y_{l, \{m\}}$ is

$$d_{[l]}(SO(D)) = (D + 2l - 2) \frac{(D + l - 3)!}{l!(D - 2)!}. \quad (32)$$

Its Casimir is $\sum_{i < j} L_{ij}^2 = l(l + D - 2)\hbar^2$, where the orbital angular momenta are defined as $L_{ij} = r_i p_j - r_j p_i$.

We also need to employ the Γ -matrices. The 2×2 Pauli matrices are just the rank-1 Γ -matrices. They can be generalized to rank- k Γ -matrices which contains $2k + 1$ matrices anti-commuting with each other. Their dimensions are $2^k \times 2^k$. A convenient recursive definition is constructed based on the rank- $(k - 1)$ Γ -matrices as

$$\begin{aligned} \Gamma_i^{(k)} &= \begin{bmatrix} 0 & \Gamma_a^{(k-1)} \\ \Gamma_a^{(k-1)} & 0 \end{bmatrix}, \quad \Gamma_{2k}^{(k)} = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix}, \\ \Gamma_{2k+1}^{(k)} &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \end{aligned} \quad (33)$$

where $i = 1, \dots, 2k - 1$. For $D = 2k + 1$ -dimensional space, its fundamental spinor is 2^k -dimensional. The generators are constructed $S_{ij} = \frac{1}{2} \Gamma_{ij}^{(k)}$ where

$$\Gamma_{ij}^{(k)} = -\frac{i}{2} [\Gamma_i^{(k)}, \Gamma_j^{(k)}]. \quad (34)$$

For the $D = 2k$ -dimensional space, there are two irreducible fundamental representations with 2^{k-1} components. Their generators are denoted as S_{ij} and S'_{ij} , respectively, which can be constructed based on both rank- $(k - 1)$ $\Gamma_i^{(k-1)}$ and $\Gamma_{ij}^{(k-1)}$ -matrices. For the first $2k - 1$

dimensions, the generators share the same form as

$$S_{ij} = S'_{ij} = \frac{1}{2}\Gamma_{ij}^{(k-1)}, \quad 1 \leq i < j \leq 2k-1, \quad (35)$$

while other generators $S_{i,2k}$ and $S'_{i,2k}$ differ by a sign as

$$S_{i,2k} = S'_{i,2k} = \pm \frac{1}{2}\Gamma_i^{(k-1)}, \quad 1 \leq i \leq 2k-1. \quad (36)$$

We couple $Y_{l,\{j_m\}}$ to the $SO(D)$ fundamental irreducible spinors. For simplicity, we use the same symbol s in this paragraph to denote the fundamental spinor representation (Rep.) for $SO(D)$ with $D = 2k+1$ and the two irreducible spinor Reps. for $SO(D)$ with $D = 2k$. The states split into the positive (j_+) and negative (j_-) helicity sectors. The bases are expressed as $\mathcal{Y}_{j_\pm; s, l, \{j_m\}}(\hat{\Omega})$, where $\{j_m\}$ is a set of $D-2$ quantum numbers for the subgroup chain. The degeneracy number of $\mathcal{Y}_{j_+; s, l, \{j_m\}}(\hat{\Omega})$ is

$$d_{j_+} = d_s \frac{(D+l-2)!}{l!(D-2)!}, \quad (37)$$

where d_s is the dimension of the fundamental spinor representation. Similarly, the degeneracy number of $\mathcal{Y}_{j_-; s, l, \{j_m\}}(\hat{\Omega})$ is

$$d_{j_-} = d_s \frac{(D+l-3)!}{(l-1)!(D-2)!}. \quad (38)$$

The eigenvalues of the spin-orbit coupling term $\sum_{i < j} \Gamma_{ij} L_{ij}$ for the sectors of $\mathcal{Y}_{j_+; s, l, \{j_m\}}(\hat{\Omega})$ and $\mathcal{Y}_{j_-; s, l, \{j_m\}}(\hat{\Omega})$ are $\hbar l$ and $-(l+D-2)\hbar$, respectively. We present the eigenstates of the D -dimensional harmonic oscillator with fundamental spinors in the total angular momentum basis as

$$\psi_{n_r, j_\pm, s, l, \{j_m\}}^D(\vec{r}) = R_{n_r, l}(r) \mathcal{Y}_{j_\pm; s, l, \{j_m\}}(\hat{\Omega}), \quad (39)$$

where the radial wavefunction reads

$$R_{n_r, l}(r) = N_{n_r, l}^D \left(\frac{r}{l_0}\right)^l e^{-\frac{r^2}{2l_0^2}} F(-n_r, l + \frac{D}{2}, \frac{r^2}{l_0^2}) \quad (40)$$

and the normalization constant reads

$$N_{n_r, l}^D = \frac{l_0^{-\frac{D}{2}}}{\Gamma(l + \frac{1}{2}D)} \sqrt{\frac{2\Gamma(n_r + l + \frac{D}{2})}{\Gamma(n_r + 1)}}. \quad (41)$$

V. THE LLS OF ODD DIMENSIONAL DIRAC FERMIONS

In this section, we generalize the 3D LL Hamiltonian for Dirac fermions to an arbitrary odd spatial dimensions $D = 2k+1$. We need to use the rank- k Γ -matrices, which contains $2k+1$ anti-commutable matrices at the dimensions of $2^k \times 2^k$ denoted as $\Gamma_i^{(k)}$ ($1 \leq i \leq 2k+1$). The definition of $\Gamma_i^{(k)}$ and the background information of

the representation of the $SO(D)$ group is given in Sect. IV.

We define $\hat{B}_{2k+1} = -i\Gamma_i^{(k)}\hat{a}_i$ and the $2k+1$ -dimensional LL Hamiltonian of Dirac fermions H_{2k+1} in the same way as in Eq. 7. Again the square of H_{2k+1} reduces to a block-diagonal form as

$$\begin{aligned} \frac{(H_{2k+1})^2}{\frac{1}{2}\hbar\omega} &= \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 - \omega \left\{ \sum_{i < j} L_{ij} \Gamma_{ij}^{(k)} \right. \\ &\quad \left. + \frac{2k+1}{2}\hbar \right\} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \end{aligned} \quad (42)$$

where $\Gamma_{ij}^{(k)} = -\frac{i}{2}[\Gamma_i^{(k)}, \Gamma_j^{(k)}]$. Each diagonal block of Eq. 42 is just the form the $2k+1$ D LL problem of non-relativistic fermions in Ref. [1].

Again we can define the following conserved quantity

$$K = \begin{bmatrix} \Gamma_{ij}^{(k)} L_{ij} + (D-2)\hbar & 0 \\ 0 & -(\Gamma_{ij} L_{ij} + \hbar) \end{bmatrix}, \quad (43)$$

K divides the eigenstates into two sectors $\Psi_{\pm n_r, j, \{j_m\}}^I$ and $\Psi_{\pm n_r, j, \{j_m\}}^{II}$,

$$\begin{aligned} K \Psi_{\pm n_r, j, \{j_m\}}^I &= \hbar(l+D-2) \Psi_{\pm n_r, j, \{j_m\}}^I, \\ K \Psi_{\pm n_r, j, \{j_m\}}^{II} &= -\hbar l \Psi_{\pm n_r, j, \{j_m\}}^{II}, \end{aligned} \quad (44)$$

respectively. As explained in Sect. IV, j represents the spin-orbit coupled representation for the $SO(D = 2k+1)$ group, and $\{j_m\}$ represents a set of good quantum number of the subgroup chain from $SO(D-1)$ down to $SO(2)$.

Similarly as before, the sectors of $\Psi_{\pm n_r, j, \{j_m\}}^{I, II}$ are dispersionless and dispersive with respect to j , respectively. The concrete wavefunctions are the same as those in Eq. 25 and Eq. 27 by replacing the 3D wavefunction to the D -dimensional version of Eq. 39. The wavefunctions of $\Psi_{\pm n_r, j, \{j_m\}}^{I, II}$ are given explicitly as

$$\begin{aligned} \Psi_{\pm n_r, j, j_z}^I(\vec{r}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{n_r, s, j_+, l, \{j_m\}}^D(\vec{r}) \\ \pm i \psi_{n_r-1, s, j_-, l+1, \{j_m\}}^D(\vec{r}) \end{bmatrix}, \\ \Psi_{\pm n_r, j, j_z}^{II}(\vec{r}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mp i \psi_{n_r, s, j_-, l+1, \{j_m\}}^D(\vec{r}) \\ \psi_{n_r, s, j_+, l, \{j_m\}}^D(\vec{r}) \end{bmatrix}. \end{aligned} \quad (45)$$

The dispersion relation for the LL branch of $\Psi_{\pm n_r, j, \{j_m\}}^I$ still behaves as $E_{\pm n_r, j, \{j_z\}} = \pm \hbar\omega \sqrt{n_r}$, while that for the branch of $\Psi_{\pm n_r, j, \{j_m\}}^{II}$ reads as $E_{\pm n_r, j, \{j_z\}} = \pm \hbar\omega \sqrt{n_r + l + \frac{D}{2}}$.

Again each occupied zero energy LL contributes to $\frac{1}{2}$ -fermion vacuum charge. If the zeroth LLs are fully filled, the vacuum charge is still $\rho^D(\vec{r}) = \frac{1}{2} \sum_{j, \{j_m\}} |\Psi_{0, j, \{j_m\}}^I(\vec{r})|^2$, which is expressed as

$$\rho^D(r) = \frac{1}{l_0^D} \left\{ \sum_{l=0}^{\infty} \frac{1}{\Gamma(l + \frac{D}{2})} \left(\frac{r}{l_0}\right)^{2l} \frac{g_l(D)}{\Omega_D} \right\} e^{-\frac{r^2}{l_0^2}}, \quad (46)$$

where $D = 2k + 1$; $g_l(D)$ is the degeneracy of the positive helicity sector of the fundamental spinor coupling to the l -th D -dimensional spherical harmonics, and its expression is the same as d_{j+} given in Eq. 37; $\Omega_D = D\pi^{D/2}/\Gamma(D/2 + 1)$ is the area of D -dimensional unit sphere. Eq. 46 can be summed analytically as

$$\begin{aligned} \rho(r) &= \frac{\sqrt{2}}{4} \left(\frac{2}{\pi l_0^2} \right)^{\frac{D}{2}} F(D-1, \frac{D}{2}, \frac{r^2}{l_0^2}) e^{-\frac{r^2}{l_0^2}} \\ &\rightarrow \frac{1}{(2\pi)^{\frac{D-1}{2}} l_0^D} \frac{1}{\Gamma(\frac{D-1}{2})} \left(\frac{r}{l_0} \right)^{D-2}, \text{ as } r \rightarrow \infty. \end{aligned} \quad (47)$$

Similarly, if the D -dimensional version of the mass term inside Eq. 9 is added, every wavefunction with the radial quantum number n_r hybridizes with its partner with $-n_r$ while keeping all other quantum numbers the same. The pair of new eigenvalues becomes $\pm \sqrt{(\hbar\omega)^2 n_r + \Delta^2}$. Again, the zero-th LL wavefunctions single out and remain the same, but their energies are shifted to Δ . For a similar open surface problem to that in Sect. III C, each LL with $n_r \neq 0$ develops a branch of gapless surface mode with the upturn (downturn) dispersion with respect to j for $n_r > 0$ ($n_r < 0$), respectively. The surface mode from the zeroth LL develops either upturn or downturn dispersions depending on the relative sign of the background field coupling and the vacuum mass.

VI. THE LLS OF EVEN DIMENSIONAL DIRAC FERMIONS

The LL problem in the even dimensions with $D = 2k$ is more complicated. The $SO(2k)$ group has two irreducible fundamental spinor representations s and s' . Each of them is with the dimension of 2^{k-1} . The construction of the $SO(2k)$ generators for the irreducible representations are introduced in Sect. IV.

Now we define $\hat{B}_{2k} = -i\Gamma_i^{(k)} \hat{a}_i$ where i runs over 1 to $2k$. Similarly to the 2D case, the counterpart of Eq. 7 in the $D = 2k$ dimensions $H_{D=2k}$ is reducible into two $2^k \times 2^k$ blocks as

$$H_{\pm} = \frac{\hbar\omega}{2} \begin{bmatrix} 0 & \pm \hat{a}_{2k}^{\dagger} + i\Gamma_i^{(k-1)} \hat{a}_i^{\dagger} \\ \pm \hat{a}_{2k} - i\Gamma_i^{(k-1)} \hat{a}_i & 0 \end{bmatrix}, \quad (48)$$

where the repeated index i runs over from 1 to $2k-1$. For each one of the reduced Hamiltonians H_{\pm} , each off-diagonal block has only the $SO(2k-1)$ symmetry. Nevertheless, each of H_{\pm} is still $SO(2k)$ invariant. If we combine the two irreducible fundamental spinor representations s and s' together, the spin generators are defined as

$$S_{ij;s \oplus s'} = -\frac{i}{4} [\Gamma_i^{(k)}, \Gamma_j^{(k)}]. \quad (49)$$

Both of H_{\pm} commute with the total angular momentum operators in the combined representation of $s \oplus s'$ defined as

$$J_{ij;s \oplus s'} = L_{ij} + S_{ij;s \oplus s'}. \quad (50)$$

We choose H_+ as an example to present the solutions of the LL wavefunctions in even dimensions. The K_+ operator is similarly defined as in Eq. 43 as

$$K_+ = \begin{bmatrix} 2S_{ij}L_{ij} + (D-2)\hbar & 0 \\ 0 & -(2S'_{ij}L_{ij} + \hbar) \end{bmatrix}, \quad (51)$$

where i, j run from 1 to $2k$, and S_{ij} and S'_{ij} are generators in the two fundamental spinor representations given in Eqs. 35 and 36, respectively. They again can be divided into two sectors of $\Psi^{+,I}$ and $\Psi^{+,II}$ whose eigenvalues of K_+ are $\hbar(l + D - 2)$ and $-\hbar l$, respectively. The dispersionless branch of $\Psi^{+,I}$ can be viewed as LLs, whose wavefunctions read

$$\Psi_{\pm n_r, j, \{j_m\}}^{+,I}(\vec{r}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{n_r, j+, s, l, \{j_m\}}(\vec{r}) \\ \mp i \psi_{n_r-1, j-, s', l+1, \{j_m\}}(\vec{r}) \end{bmatrix}. \quad (52)$$

Their spectra are same as before $E_{\pm n_r, j, \{j_m\}} = \pm \hbar\omega \sqrt{n_r}$. Please note that the upper and lower components involve the s and s' representations, respectively. Similarly, the dispersive solutions of Ψ^{II} become

$$\Psi_{\pm n_r, j, \{j_m\}}^{+,II}(\vec{r}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mp i \psi_{n_r, j-, s, l+1, \{j_m\}}(\vec{r}) \\ \psi_{n_r, j+, s', l, \{j_m\}}(\vec{r}) \end{bmatrix}, \quad (53)$$

whose dispersions read $E_{\pm n_r, j, \{j_m\}} = \pm \hbar\omega \sqrt{n_r + l + \frac{D}{2}}$. The solutions to H_- are very similar to Eq. 52 and Eq. 53 by exchanging the irreducible fundamental spinor representation indices s and s' .

Again if the zeroth branch LLs are filled, the vacuum charge $\rho^D(\vec{r}) = \frac{1}{2} \sum_{j, \{j_m\}} |\Psi_{0, j, \{j_m\}}^I(\vec{r})|^2$ is calculated as

$$\rho^D(r) = \frac{1}{l_0^D} \left\{ \sum_{l=0}^{\infty} \frac{1}{\Gamma(l + \frac{D}{2})} \left(\frac{r}{l_0} \right)^{2l} \frac{g_l(D)}{\Omega_D} \right\} e^{-\frac{r^2}{l_0^2}}, \quad (54)$$

where $D = 2k$, Ω_D and $g_l(D)$ are defined similarly as in Eq. 46. It can be summed over analytically as

$$\rho^D(r) = \frac{1}{4} \left(\frac{2}{\pi l_0^2} \right)^{\frac{D}{2}} F(D-1, \frac{D}{2}, \frac{r^2}{l_0^2}) e^{-\frac{r^2}{l_0^2}}, \quad (55)$$

which $\rho^D(r) \rightarrow \frac{\sqrt{\pi}}{(2\pi l_0^2)^{\frac{D}{2}}} \frac{1}{\Gamma(\frac{D-1}{2})} \left(\frac{r}{l_0} \right)^{D-2}$, as $r \rightarrow +\infty$.

VII. CONCLUSION

In summary, we have generalized the LL problem of 2D Dirac fermions to arbitrary higher dimensional flat spaces with spherical symmetry. This problem is essentially the square root problem of its non-relativistic LL problem with spherical symmetry in high dimensions Ref. 1. The

zero energy LLs is a branch of $\frac{1}{2}$ -fermion modes. On the open boundary, each LL contributes one branch of helical surface modes. This series of LL problems can be viewed as the generalization of parity anomaly in 2D to arbitrary dimensions in a spherical way. An open question is that how to experimentally realize the case of the 3D systems.

Note added Near the completion of this manuscript, we became aware of that Eq. 8 plus Eq. 9 with a special relation between the background field coupling constant

and the mass term was studied in Ref. [41,42] under the name of Dirac oscillator. We have studied a more general form from a different perspective and identified their relation to the LLs.

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